approximate expression

$$y(x,\eta)\approx\eta+\frac{aS-bQ}{\sqrt{\lambda_l}\,A_{ll}}\sin\sqrt{\lambda_l}\,x\varphi_l(\eta) \tag{5.6}$$

$$a=-\nu_l\int_{\eta_0}^{1}\varphi_l(\eta)\,d\mu(\eta),\quad b=-\varphi_l(\eta_0)$$

As $\nu\to\nu_l+0$, a quantity of the order of

$$O(\exp(-\sqrt{|\lambda_{N+1}|}\,(x-l)))+o(\sqrt{(S^2+Q^2)/\lambda_l})$$

is the estimate in (5.6).

For $|x|<l$, series (4.4) converges in the mean-square sense. If we use the asymptotic properties of the eigenvalues λ_m and the eigenfunctions $z_m(\eta,\nu)$ as $m\to+\infty$, then the standard technique of mathematical physics enables us to separate the singularities at the body boundary and to improve the convergence of the series. A more detailed study of the near velocity field would make it possible to determine the distortions to the body shape by the hypothesis of the possibility of approximating the velocity profile at the body boundary by that of a weightless fluid flowing past. This question is not discussed in the present paper. Here we merely remark that in the arrangement discussed the streamline which corresponds to the body remains closed, and the area bounded by this streamline equals the area of the body.

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ANISOTROPIC TURBULENCE IN THE FLOW OF AN INCOMPRESSIBLE FLUID BETWEEN PARALLEL PLANE WALLS*

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It is shown that in the region adjacent to a solid wall a Newtonian fluid in turbulent flow can be regarded as an oriented Ericsson-Leslie fluid whose defining constants are subject to certain conditions. The logarithmic velocity profile is obtained from the solution found if the molecular viscosity is ignored, when the distance from the wall is small.

1. Consider the confined turbulent flow of an incompressible Newtonian fluid between plane parallel walls in the absence of mass forces. The coordinate system consists of an x-axis directed along the flow, and a y-axis perpendicular to the walls. The wall equation is $y=\pm h$.

The Prandtl semi-empirical theory of the mixing length, and numerous experiments show that in the vicinity of a solid wall the longitudinal averaged velocity u has the following logarithmic profile:

$$\frac{u}{v_*}=\frac{1}{\varkappa}\ln\left(1-\frac{|y|}{h}\right)+C \tag{1.1}$$

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where $v_* = (\tau_w / \rho)^{1/2}$ is the dynamic velocity, τ_w is the modulus of the tangential stress an the wall, ρ is the density of the fluid, and $\varkappa$ is Karman's constant.

It will be clear from what follows that the profile (1.1) is obtained if the turbulent fluid is regarded as an anisotropic fluid whose physical properties are defined by an orientation vector $\mathbf{n}$, see /1 - 3/.

In a turbulent flow, the Newtonian fluid becomes anisotropic (/4–6/), and close to a solid wall it has a structure formed by the system of vortices referred to as Λ-vortices, /6/. The Λ-vortex has an apex which is the point most distant from the wall, and two branches which run to infinity in the direction of the flow. As the distance between the branches and the apex increases, they approach the wall, and the angle between them and the flow direction tends to zero. Although some vortices may depart from the wall, an overwhelming number are situated in the wall region, which we denote by δ.

We will use the Ericsson-Leslie model to describe the turbulent fluid /1 - 3/, assuming that the unit vector $\mathbf{n}$ describes the direction of the vortex line. Although this model was constructed for nematic crystals, it can be seen from the structures of the relevant equations that it can be used for other fluid media whose properties are characterized by the orientation vector.

Suppose that we are given the steady flow of an incompressible fluid where the body forces can be ignored. Assuming that

$$v_x = u\,(y),\ v_y = v_z = 0 \tag{1.2}$$

$$\mathbf{n} = \{\cos\theta\,(y),\ \sin\theta\,(y),\ 0\} \tag{1.3}$$

we can obtain (see /1/) the following equations of motion of an oriented fluid between parallel plane walls (the prime denotes differentiation with respect to y):

$$\tau_{xy}' = \partial p / \partial x, \quad \tau_{yy}' = 0 \tag{1.4}$$

$$\mu_{xy}' + g_x = 0, \quad \mu_{yy}' + g_y = 0 \tag{1.5}$$

Here $v_x,\ v_y,\ v_z$ are the components of the averaged velocity, $\tau_{ij}\ (i,\ j = x,\ y,\ z)$ are the tensions, p is the pressure, and θ is the angle between the reference vector and the x-axis. The generalized tensions μ_{ij} and the inner volumetric force $\mathbf{g}\ (g_x,\ g_y,\ g_z)$ are dynamic quantities characteristic for an oriented fluid, which define the variation of the orientation vector /1, 2/.

As a consequence of (1.2), the discontinuity equation holds identically. The quantities in (1.4) and (1.5) are determined by the following formulae:

$$\begin{aligned}
2\tau_{xy} &= [2\mu_1 \sin^2\theta \cos^2\theta + (\mu_5 - \mu_2)\sin^2\theta + (\mu_3 + \mu_6)\cos^2\theta + \\
&\quad \mu_4]\, u' \\
\tau_{yy} &= -p - (k_{11}\cos^2\theta - k_{33}\sin^2\theta)\,\theta'^2 - \sin\theta\cos\theta\,[\mu_1 \sin^2\theta + \\
&\quad {}^1/_2\,(\mu_2 + \mu_3 + \mu_5 + \mu_6)]\, u' \\
\mu_{xy} &= \beta_2 \cos\theta - [k_{22}\sin\theta + (k_{33} - k_{22})\sin^3\theta]\,\theta' \\
\mu_{yy} &= \beta_2 \sin\theta + [k_{11}\cos\theta + (k_{33} - k_{22})\cos\theta\sin^2\theta]\,\theta' \\
g_x &= \gamma\cos\theta + \beta_2\theta'\sin\theta + {}^1/_2\,(\lambda_2 - \lambda_1)\, u'\sin\theta \\
g_y &= \gamma\sin\theta - \beta_2\theta'\cos\theta - (k_{33} - k_{22})\,\theta'^2\sin\theta + {}^1/_2\,(\lambda_1 + \lambda_2)\, u'\cos\theta
\end{aligned} \tag{1.6}$$

where $\lambda_1,\ \lambda_2,\ \mu_1, \ldots, \mu_6,\ k_{11},\ k_{22},\ k_{33}$ are the defining constants in the Ericsson-Leslie model, and β_2 and γ are undetermined constants in the model.

The constants of the Ericsson-Leslie model, in particular those which occur in Eqs.(1.6), are connected by the following relations (see /2/):

$$\begin{aligned}
& k_{11} \geqslant 0,\ k_{22} \geqslant 0,\ k_{33} \geqslant 0,\ |k_{24}| \leqslant k_{22},\ \ |k_{11} - k_{22} - k_{24}| \leqslant k_{11} \\
& \mu_2 + \mu_3 = \mu_6 - \mu_5,\ \ \mu_4 \geqslant 0,\ \ 2\mu_1 + 3\mu_4 + 2\mu_5 + 2\mu_6 \geqslant 0, \\
& 2\mu_4 + \mu_5 + \mu_6 \geqslant 0,\ -4\lambda_1\,(2\mu_4 + \mu_5 + \mu_6) \geqslant (\mu_2 + \mu_3 - \lambda_2)^2 \\
& \lambda_1 = \mu_2 - \mu_3,\ \ \lambda_2 = \mu_5 - \mu_6
\end{aligned} \tag{1.7}$$

On substituting into Eqs.(1.5) the last four expressions in (1.6), we obtain the equation which is satisfied by the angle θ,

$$\begin{aligned}
& d/d\theta\,(f\,(\theta)\,\theta'^2) + (\lambda_1 + \lambda_2\cos 2\theta)\, u' = 0 \\
& f\,(\theta) = k_{11}\cos^2\theta - k_{33}\sin^2\theta
\end{aligned} \tag{1.8}$$

Let us find the conditions which, in addition to (1.7), should be satisfied by the constants of the Ericsson-Leslie model in order that the velocity profile in the vicinity of the wall should have the form (1.1). As in the Prandtl theory of the mixing length, we assume that next to the wall, $\tau_{xy} = -\tau_w$. Then the first equation in (1.6) takes the form

$$g(\theta)\, u' = -\tau_w \tag{1.9}$$

$$2g(\theta) = 2\mu_1 \sin^2\theta \cos^2\theta + (\mu_5 - \mu_2)\sin^2\theta + (\mu_3 + \mu_6)\cos^2\theta + \mu_4 \tag{1.10}$$

Eqs.(1.8) and (1.9) give the following differential equations for determining the angle θ:

$$\frac{d}{d\theta}\left(f(\theta)\theta'^2\right) = \frac{\lambda_1 + \lambda_2 \cos 2\theta}{g(\theta)}\tau_w \tag{1.11}$$

If the velocity u is defined by Eq.(1.1), on finding u' from (1.1) and substituting it into (1.9), we obtain an algebraic equation for determining θ,

$$g(\theta) = \rho\varkappa v_* (h - |y|)\,\mathrm{sgn}\, y \tag{1.12}$$

By its nature, (1.12) is obviously the integral of equation (1.11). By Eq.(1.12),

$$\theta' = \frac{2\rho\varkappa v_*}{(-2\mu_1\cos 2\theta + \mu_2 + \mu_3 - \mu_5 + \mu_6)\sin 2\theta} \tag{1.13}$$

If we substitute the derivative (1.13) into Eq.(1.11), then clearly the left-hand side of it will be an odd function, and the right-hand side an even function of θ. Thus, to transform Eq.(1.11) into an identity when (1.13) is substituted, it is necessary and sufficient that both sides vanish. Hence

$$k_{11} = k_{33} = 0,\ \lambda_1 = \lambda_2 = 0 \tag{1.14}$$

By Eq.(1.14), relations (1.7) give

$$\mu_2 = \mu_3 = 0,\ \ \mu_5 = \mu_6,\ \ k_{24} = -k_{22} \tag{1.15}$$

Eqs.(1.14) and (1.15) establish those constraints which must necessarily be satisfied by the coefficients of the defining equations so that the velcoity profile is logarithmic. As a model of a turbulent fluid we take the Ericsson-Leslie model of the oriented fluid (see /1, 2/) in which the constants satisfy conditions (1.14) and (1.15). In this model, taking into account Eqs.(1.2) and (1.3) we express the tensions in the flow moving between the plane parallel walls, in the form

$$\begin{aligned}
\tau_{xx} &= -p + u'(\mu_1\cos^2\theta + \mu_5)\sin\theta\cos\theta \\
\tau_{yy} &= -p + u'(\mu_1\sin^2\theta + \mu_5)\sin\theta\cos\theta \\
\tau_{xy} &= \tau_{yx} = u'[\mu_1\sin^2\theta\cos^2\theta - {}^1\!/_2(\mu_4 + \mu_5)] \\
\tau_{zz} &= -p,\ \ \tau_{xz} = \tau_{zx} = \tau_{yz} = \tau_{zy} = 0
\end{aligned} \tag{1.16}$$

As in /5/, the the tensions $\tau_{xx}, \tau_{yy}, \tau_{zz}$ are all different.

2. Let us find the velocity profile for a turbulent flow between plane parallel walls. For this, we shall solve the system of Eqs.(1.4)-(1.6) taking into account conditions (1.14) and (1.15). The system reduces to the following equations:

$$g(\theta)\, u' = -\tau_w y/h,\ \tau_w = -h\partial p/\partial x \tag{2.1}$$

$$k_{22}\sin\theta\cos\theta\theta'' + k_{22}(1 - 3\sin^2\theta)\theta'^2 - \gamma = 0 \tag{2.2}$$

$$g(\theta) = \mu_1\sin^2\theta\cos^2\theta + \mu_0,\ 2\mu_0 = \mu_4 + \mu_5 \tag{2.3}$$

We formulate the boundary conditions for the function $\theta(y)$ in conformity with /6/:

$$\theta(h) = \theta(-h) = 0 \tag{2.4}$$

The velocity $u(y)$ should satisfy the condition of adhesion at the wall

$$u(h) = u(-h) = 0 \tag{2.5}$$

If $\gamma = 0$, integrating Eq.(2.2) once we obtain

$$\sin^2\theta\cos^4\theta\theta'^2 = B \tag{2.6}$$

where B is the integration constant. If θ_0 is the angle of inclination of the orientation vector at the top boundary $y = \pm(h - \delta)$, which is adjacent to the walls of the Λ-vortex layer of thickness δ (see /6/), and θ_0' is the corresponding derivative, then

$$B = \theta_0'^2 \sin^2\theta_0\cos^4\theta_0 \tag{2.7}$$

Since the flow is symmetrical about the plane $y = 0$, we shall confine ourselves to examining the flow in the upper half-plane: $0 \leqslant y \leqslant h$. Then integration of (2.6), with boundary conditions (2.4), yields

$$\cos\theta = t, \quad t = [1 - 3\sqrt{B}\,(h - y)]^{1/2} \tag{2.8}$$

The integration of (2.1) where the function $g(\theta)$ is defined by formula (2.3) results in the following velocity profile:

$$\frac{3\mu_1 hB}{\sqrt{\rho\tau_w}}\frac{u}{v_*} = \frac{3h\sqrt{B}-1}{2q^2-1}\left[\sqrt{q^2-1}\,\mathrm{arctg}\frac{t}{\sqrt{q^2-1}} + \frac{q}{2}\ln\frac{q-t}{q+t}\right] + \frac{1+2\alpha}{4\sqrt{1+4\alpha}}\ln\frac{q^2-t^2}{t^2+q^2-1} + \frac{1}{4}\ln|t^4 - t^2 - \alpha| + \frac{t^2}{2} + C \tag{2.9}$$

$$(\alpha = \mu_0/\mu_1,\ 2q^2 = 1 + \sqrt{1+4\alpha},\ q > 0)$$

The constant C is found from the boundary condition (2.5)

$$C = \frac{1-3h\sqrt{B}}{2q^2-1}\left[\sqrt{q^2-1}\,\mathrm{arctg}\frac{1}{\sqrt{q^2-1}} + \frac{q}{2}\ln\frac{q-1}{q+1}\right] - \frac{1+2\alpha}{4\sqrt{1+4\alpha}}\ln\left(1-\frac{1}{q^2}\right) - \frac{\ln\alpha}{4} - \frac{1}{2} \tag{2.10}$$

Formulae (2.9) and (2.10) give the desired velocity distribution for the above model of a turbulent flow in the segment $(h-\delta) \leqslant y \leqslant h$. We will show that the logarithmic velocity profile (1.1) is obtained from this equation if we make the assumption which is usually made in deriving formula (1.1) (see /7/).

The function $g(\theta)$ represents the effective viscosity of a turbulent fluid, and at the same time, $\mu_0 = g(0)$. Thus, μ_0 is the viscosity directly at the wall and is essentially the coefficient of molecular viscosity. If we ignore the viscosity μ_0 as compared with the turbulent viscosity $\mu_1 \sin^2\theta\cos^2\theta$ (this means that $\alpha = 0$ and $q = 1$), then in the vicinity of the wall, where the differences $h - y$ and $1 - t$ are small and, therefore, the main term in formula (2.9) is $\ln(1-t)$, the velocity profile takes the form (1.1), where

$$\varkappa = 2\mu_1 (B/(\rho\tau_w))^{1/2} \tag{2.11}$$

(it is of interest to note that in this case the boundary condition (2.5) cannot any longer be satisfied, and the constant C can be found by introducing the concept of a laminar sublayer, see /7/).

Thus, Karman's constant $\varkappa$ proves to be connected with the parameters of the turbulent flow.

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